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The Sledge on the Slope
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Energy in the Economy, and the Paradox of Theory and Policy

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Abstract

Energy conversion in the production of goods and services, and the resulting emissions associated with entropy production, have not yet been taken into account by the mainstream theory of economic growth. Novel econometric analyses, however, have revealed energy as a production factor whose output elasticity, which measures its productive power, is much higher than its share in total factor cost. This, although being at variance with the notion of orthodox economics, is supported by the standard maximization of profit or time-integrated utility, if one takes technological constraints on capital, labor, and energy into account. The present paper offers an explanation of these findings in the picture of a sledge, which represents the economy, on the slope of a niveous mountain, which represents cost. Historical economic trajectories indicate that the representative entrepreneur at the controls of the sledge steers his vehicle with due regard of the barriers from the technological constraints, observing "soft" constraints, like the legal framework of the market, in addition. We believe that this perspective contributes to resolving the paradox that energy hardly matters in mainstream growth theory, whereas it is an issue of growing importance in international policy.

Keywords: energy, economic growth, oil price, profit maximization, technological constraints, output elasticities
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1 Introduction

Greenhouse-gas emissions must be reduced drastically, by up to 80% until the year 2050, if the increase of the surface temperature of the earth should not exceed 2 °C. This is consensus among practically all nations, by now. However, an agreement on internationally binding emission limits is difficult to reach, because important actors like the USA, China, and India fear that emisson reductions will restrict energy utilization – which causes about 65% of greenhouse-gas emissions [1] – and thus threaten economic growth. Furthermore, economic growth had suffered in the past during the first and the second oil price explosions, shown in Fig. 1, and the oil price increases since 2005 may have contributed to the economic crises since 2007 [2]. On the other hand, mainstream economists disregard energy as a factor of production. In their view, only capital and labor matter for the generation of goods and services, which make up the output of an economy,[1] and technological progress, like "mannah from Heaven", takes care of the rest. This rest, the notorious "Solow residual", exceeds 50 percent of gross domestic product in many highly industrialized economies.

If, occasionally, energy is taken into account, e.g. in the economics of climate change [3], one concludes from a special type of economic equilibrium that energy's economic weight, called output elasticity, should be equal to its share in total factor cost. This share has been roughly a meager 5 percent in major industrial countries[2] during the last five decades, while labor gets the lion's share of about 65 percent and capital 30 percent. Consequently, energy plays no, or only a minor role in the mainstream theory of economic growth, whose adherents also firmly believe in complete factor substitutability.[3] Thus, we have the paradoxical situation that the USA, homeland of most of the Nobel laureates in economics, strongly opposes internationally binding agreements on the reduction of emissions from the combustion of fossil fuels, fearing that reductions of this combustion, and of the energy supply that results from it, will endanger economic growth, while at the same time economic theory says that energy hardly matters for growth.[4]

Two solutions of the paradox are conceivable. Either the US negotiators at the UN conferences on climate change behave irrationally, or mainstream economists' view of energy is wrong, climate stabilization and economic growth *are* competing objectives under the fossil-fuel regime without carbon capture and storage, and the hard negotiations

[1] Typically, the Nobel laureate in economics and father of standard growth theory, Robert A. Solow, stated: "The world can, in effect, get along without natural resources", although he cautioned that "if real output per unit of resources is effectively bounded – cannot exceed some upper limit of productivity which is not far from where we are now – then catastrophe is unavoidable". [4]

[2] The "Tranche I Taxation Study" for the UN Framework Convention on Climate Change states: "...energy expenditures amount to a relatively low percentage of Gross Domestic Product within OECD countries (between three and 11 % on a purchasing power parity basis with a 5.8% average for OECD as a whole)..."[5].

[3] During an international conference on natural resources, a young economist gave a talk on energy in the economy [6]. He explained that, because of the first law of thermodynamics, it is impossible to substitute capital for energy completely. A world-famous mathematical economist jumped up, interrupted him, and irate he shouted: "You must never say that! There is always a way for substitution!" N.B.: The standard Cobb-Douglas production function with energy as third input besides capital and labor assumes (asymptotically) complete factor substitutability, i.e. it includes the possibility of producing a certain output with arbitrarily small energy input, if capital or labor are increased sufficiently.

[4] Thus, all efforts to secure oil supplies that go beyond those of securing the supplies of important raw materials like coltan and rare earths would be difficult to justify theoretically.

on binding emission limits do have a serious economic basis in the real world. Furthermore, if the second alternative is correct, increasing scarcity of conventional oil and gas, for the reasons pointed out by the "peak oil" theory [2, 7], will cause economic problems in the future.

In this paper, we try to explain how econometric analyses that reveal energy as an essential factor of production, can be understood by a reappraisal of economic equilibrium and of behavioral assumptions. The explanation is based on a mathematical framework that deviates from that of standard economics in the calculation of output elasticities.[5]

2 Natural laws, technological constraints, and wealth production

The real world is governed by the laws of nature. The most powerful ones are the first and the second law of thermodynamics. They say that nothing happens in the world without energy conversion and entropy production. Thus, there is no economic production and growth whatsoever without energy conversion. And entropy production, inevitably coupled to energy conversion, results in the emission of heat and particles that may change the environment, including the climate [8]. This imposes limits to growth on earth. Furthermore, entropy production reduces the potentials of energy conservation: Thermoeconomic optimization at a *fixed* demand for industrial energy services cannot reduce primary energy input by much more than 50 percent.

Therefore, economic growth faces obstacles that are rooted in "the constitution of the universe", as the first and the second of thermodynamics are sometimes called. They are ignored by standard economic theory. The first step to reconciling the science of economics with these laws is taking energy into account as a factor of production on an equal footing with capital and labor. This includes proper calculation of each production factor's output elasticity, which, roughly speaking, gives the percentage of output change when the factor changes by one percent, while the other factors stay constant. It indicates the productive power of the factor.

In standard economics the output elasticities are obtained from the conditions for the neoclassical economic equilibrium, which is computed under idealizing behavioral and technological assumptions. According to the former, the only objectives of economic actors are either the maximization of profit or of time-integrated utility. According to the latter, there are no constraints whatsoever on the combinations of capital, labor, and energy, so that the optimization calculus can determine the equilibrium in any point of positive factor space. In this equilibrium, output elasticities are equal to factor cost shares. We call this the cost share theorem. But from an engineering point of view it is obvious that there *are* technological constraints on factor combinations. They concern the capital stock, which consists of all energy conversion devices and information processors and the buildings and installations necessary for their protection and operation. They are given by the fact that i) the degree of capacity utilization cannot exceed 1 and ii) the degree of automation of the capital stock cannot exceed a time-dependent number that is less than or equal to 1. Inclusion of these constraints in the optimization calculus leads to

[5]Evolutionary economists comment that they agree with our results but dislike our methods, which, in their view, are still too close to neoclassical economics. Rather, economics should be completely based on non-equilibrium thermodynamics [9].

equilibrium conditions, where shadow prices, which translate technological contraints into monetary terms, add to factor prices [10]. If at a given time factor prices are such that the cost-minimizing equilibrium actually borders on one or even on both barriers from the technological constraints, one or even both shadow prices are non-zero, and the cost share theorem is invalid. Thus, even if one accepts the behavioral assumptions of standard economics, output elasticities are in general not equal to factor cost shares. Consequently, they have to be determined otherwise.

An alternative way of computing output elasticities, which is independent from any behavioral assumptions, has been developed since the 1980s. It is summarized in [8, 11], and is briefly sketched in the Appendix. It is based on the usual requirement that the output of an economic production system at a given time should only depend on the production factors at that time, i.e. the production function, which describes the output as a function of the inputs, must be twice differentiable. Then, at any fixed time t, a growth equation holds, which says that the growth ratio of output is the weighted sum of the growth ratios of capital, labor, and energy. The evolution of the economy can be computed by integrating the growth equation along any convenient path in the capital-labor-energy space that is accessible to the economy. The weights of the production factors in the growth equation are the output elasticities. We call them α for capital, β for labor, and $\gamma = 1 - \alpha - \beta$ for energy. They have to satisfy a set of three coupled partial differential equations, whose most general solutions are any differentiable functions of the ratios of labor to capital and of energy to capital. The special solutions, and the resulting exact production function, could be computed, if for a given economy one knew the appropriate boundary conditions of the differential equations exactly. This, however, is impossible. The reason for that is indicated in the Appendix. Therefore, all output elasticities and production functions are only approximations.

The simplest approximation are constant output elasticities α_0, β_0, and γ_0. The corresponding integral of the growth equation is the energy-dependent version of the well-known Cobb-Douglas production function. The simplest factor-dependent solutions are the output elasticities $\alpha_{L1} = a(l+e)/k$, $\beta_{L1} = a\,(cl/e - l/k)$, and $\gamma_{L1} = 1 - ae/k - acl/e$; here the dimensionless numbers k, l, and e are multiples of capital, labor, and energy in a base year t_0. They belong to the (first) LinEx production function, which depends *lin*early on energy and *ex*ponentially on factor quotients. They satisfy the *asymptotic* boundary conditions that α_{L1} should vanish for vanishing ratios of labor and energy to capital and thus reflect the law of diminishing returns, and that β_{L1} should vanish, when the capital stock approaches the magnitude k_m required for maximum automation, and when simultaneously the energy input approaches the quantity $e_m = ck_m$ that is demanded by the fully utilized capital stock k_m; furthermore, the input of (routine) labor in the state of maximum automation should be much smaller than e_m and k_m. The integration constants a (capital effectiveness) and c (energy demand of the capital stock) are technology parameters that are determined econometrically, subject to the restrictions that output elasticities must be non-negative.[6]

The LinEx function reproduces economic growth, and the recessions during the first and second oil-price explosions, in the USA, Japan, and Germany during a significant part of the 20th century with small residuals and good statistical quality measures. Its

[6]The restrictions can be handled by the Levenberg-Marquardt method. Some time depence of a and c has to be allowed, if OLS fitting of the LinEx function is done to empirical time series of output that comprise more than two decades [8].

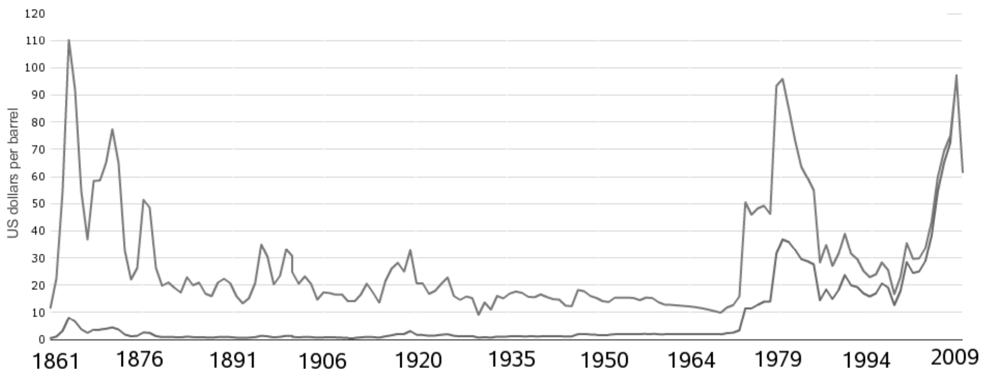

Figure 1: Development of the price of one barrel of crude oil since 1861 in 2009 US dollars, upper curve, and in dollars of the day, lower curve. Source: http://www.pdviz.com/historical-crude-oil-prices-1861-to-2009

time-averaged output elasticities, which are roughly $\bar{\alpha}_{L1} \approx 0.4$, $\bar{\beta}_{L1} \approx 0.1$, and $\bar{\gamma}_{L1} \approx 0.5$, are for energy much larger and for labor much smaller than the cost shares of these factors [8, 11, 12, 13]. This is confirmed by cointegration analysis. The energy-dependent Cobb-Douglas function with output elasticities that are close to the time-averaged LinEx output elasticities also reproduces growth, but with somewhat larger residuals and worse statistical quality measures. Energy accounts for most of the growth that standard economics attributes to technological progress and related concepts.

Orthodox economists reject this heresy. Their basic objection is: If the output elasticity of energy were much larger and that of labor were much smaller than the cost shares of these factors, profit-maximizing entrepreneurs would raise the input of energy and lower that of labor until output elasticities and cost shares would be equal. The simple, direct reply to this is: Increasing the energy input by the required amount would drive the economy right against one or both of the technological shadow-price barriers and prevent it from reaching the neoclassical equilibrium. Furthermore, awareness of the these barriers and "soft" constraints will keep the evolution of real-life economies at some distance from the barriers. Subsequently we try to describe this in the picture of a sledge on a snow-covered slope, and substantiate this picture by the quantitative analysis of a historical trajectory.

3 The economy in the cost mountain

We consider the mountain of factor cost that rises above the plane that is spanned by the input ratios "labor/capital" and "energy/capital". Snow fall, which may also trigger avalanches, and the resulting changes of the niveous topography of the mountain slope, represent changing factor prices. Shadow-price barriers that definitely block access to certain regions of the mountain represent the technological constraints. The economy is a sledge that moves on the slope. It is handled by the representative entrepreneur (REPRENT) according to his set of objectives. The behavior of this entrepreneur is the resultant from the decisions and actions of all real-life economic actors. The cost

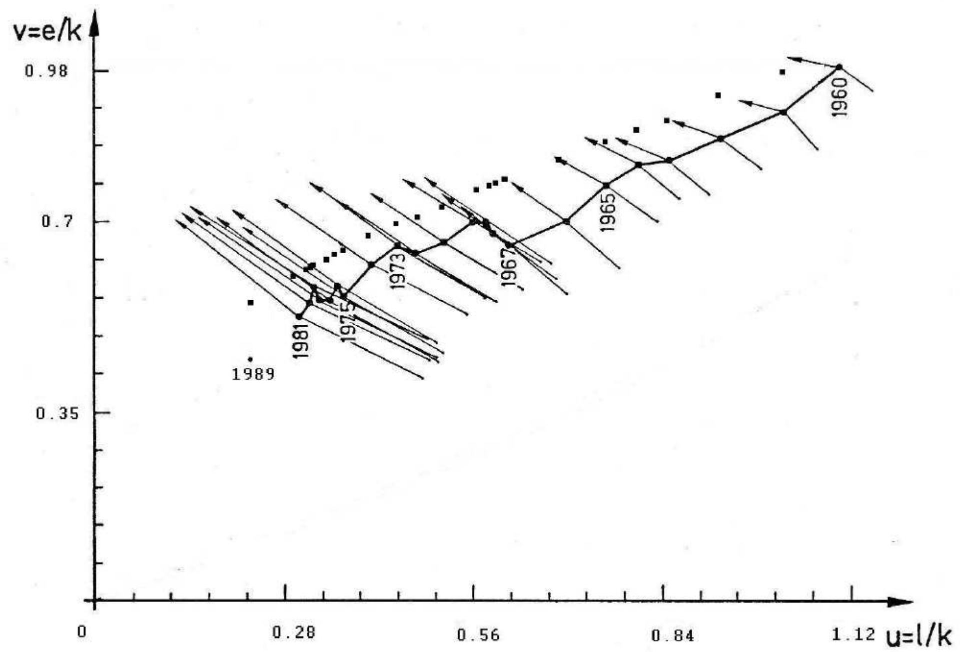

Figure 2: The solid line indicates the path of the German industrial sector "Warenproduzierendes Gewerbe" (GWG) in the cost mountain between the years 1960 and 1981, projected onto the $u - v$ plane; $u \equiv l/k$ and $v \equiv e/k$, where k, l, and e are multiples of capital, labor, and energy in the base year $t_0 = 1960$. Full squares mark the shadow-price barrier from the limit to capacity utilization. Direction and length of the arrows above the path indicate directions and strengths of the negative cost gradients, whose components are calculated with the LinEx production function; arrow lengths are for the 1970 output. Below the path, the lines without arrow heads indicate the negative cost gradients obtained with the energy-dependent Cobb-Douglas function, whose output elasticities are close to the time-averaged elasticities of the LinEx function. The gradients depend little on the type of production function. What matters is the magnitude of the output elasticities. For details see the text and the Appendix.

mountain, which represents negative profit as well, is mathematically described in the Appendix. In Fig. 2 the trajectory of the German industrial sector "Warenproduzierendes Gewerbe" (GWG) in the cost mountain and the negative cost gradients are projected onto the mountain base, as an example.[7] The considered time is that between the years 1960 and 1981, when during the first and the second oil-price explosions, shown in Fig. 1, the aggregate energy price increased in Germany by a factor of 2.5.[8] Similar price shocks hit the other industrial market economies and triggered their first two serious post-war recessions. During these recessions, the downturn of economic output closely followed the decrease of energy input [8]. Since then, energy conservation measures have been taken, and energy-intensive, polluting production processes have been shifted abroad. As a

[7]GWG is the pillar of the German economy. It produced about 50 percent of (West) German GDP in the 1960s and 1970s [8].

[8]However, the changing energy prices have little influence on the cost gradients, because energy's cost share is small, even in the oil price maxima. The numbers are given in [14].

result, the technology parameters a and c of the LinEx function acquire a time dependence, which, however, has been negligible for the time span covered by the trajectory in Fig. 2.

The general direction of this trajectory is at a large angle to the negative cost gradients, which point toward the region of small $u \equiv l/k$ and much larger $v \equiv e/k$. Only during times of economic upswings, as from 1967-1970, 1972-1973, and 1975-1979, the path follows temporarily the negative cost gradients toward the barrier from the limit to capacity utilization. This shows that increasing the energy input into the machines that were not fully used in times of recessions, rapidly increases output and profit. The isolated point, marked by 1989, indicates that, until German reunification in 1990, the general direction of the path toward strongly decreasing u and moderately decreasing v continues. It is more or less parallel to the shadow-price barrier from the limit to capacity utilization. Obviously, the pilot of the sledge, i.e. the REPRENT of the economic sector GWG, sees this barrier and steers his vehicle along it, but without touching it. This leaves maneuvering space for him in the sense that capital, labor, and energy can be chosen freely on the uphill side of the barrier. Neither does the REPRENT steer the sledge right downhill along the negative cost gradients toward the region of maximum automation, where v is close to 1 and u is very small.[9] The REPRENT is obviously aware of a much closer barrier to automation, which results from volume, mass and energy demand of the information processors in the capital stock. Imagine the vacuum-tube computers of the 1960s, when the tiny transistor, invented in 1947 by Bardeen, Brattain and Shockley, had not yet diffused into the capital stock. A vacuum-tube computer with the computing power of a 2010 notebook computer would have had a volume of many thousands of cubic meters. In the 1960s, a degree of automation, that is standard 40 years later in the highly industrialized countries, would have resulted in factories many orders of magnitude bigger than today, probably exceeding the available land area.[8] Thus, a path of economic evolution is chosen that reduces the input of expensive routine labor, substituting cheaper energy/capital combinations for it, without trying to put the economy right at a shadow-price barrier. Along this path the REPRENT satisfies the increasing demand for goods and services with decreasing energy content (i.e. more information technology and less excavation of construction sites by bulldozers, for instance), and he also uses more and more imported intermediate goods, whose embodied energies do not show in the national energy balances. Gaining a larger share of the market is also an objective that sacrifices short-term cost minimization for long-term profit gains. He also has to observe legal and social obligations that impede the firing of workers when instant cost cutting would suggest it. He may also understand the long-term benefit of keeping at least the core of a well trained, loyal workforce.

The empirical data on capital, labor, and energy, obtained from the national statistics and compiled in [8], show that until 1973 the evolution of the sectors "Industries" of the USA and Japan differs from that of Germany: Labor grows in the USA, and remains nearly constant in Japan, whereas capital and energy increase at nearly the same rate in both countries. In GWG, on the other hand, the input of energy grows much less than the capital stock, and labor always decreases. But after the first oil price shock, the growth of energy is significantly reduced in all three systems, while the growth of capital continues

[9] There, β_{L1} vanishes for $e \to e_m = ck_m$, and c is close to 1 for most of the considered time span. In the present LinEx approximation the shadow-price barrier from maximum automation connects to the restriction that β must be non-negative, i.e. $e \leq ck$ for any k, so that its shape is uncertain and its location can be only circumscribed by $v \approx 1$ and $u << 1$.

as before. In the USA and Germany the energy input oscillates in response to the oil price variations, and in Japan it nearly flattens out. Consequently, the US and Japanese path in the $u - v$ plane stays close to $v = 1$ until 1973 while u decreases, and after 1973 both u and v decrease. Then, the REPRENTs of US and Japanese industries seem to behave similar to the REPRENT of GWG.

4 Conclusions and outlook

The assumption that economic evolution is governed by unconstrained, instant profit maximization must be modified. The representative entrepreneur of real-life economic actors, who runs the economy, is aware of and stays away from the shadow-price barriers that prevent the economy from reaching the equilibrium that is characterized by the equality of output elasticities and factor cost shares at the prices of capital, labor, and energy we have known so far. He also observes the "soft" constraints from the demand of consumers and investors, and from social and legal obligations as well. These soft constraints are much harder to model mathematically than the technological constraints, but in combination with the latter they determine the sequence of operating points through which the economy actually passes. The pilot of the sledge in the cost mountain does not simply dash downhill, but rather trots along a path that is shaped by his multi-component set of constraints and objectives.

Nevertheless, the imbalance between the economic weights and the costs of labor and energy does show in the evolution of real-life economies. The path in Fig. 2 *is* a cost-reducing trajectory along which cheap, powerful energy/capital combinations of high productive power substitute for expensive labor of low productive power. This long-term trend, which is present in all highly industrialized countries, increases automation, eliminates more and more jobs for routine labor, and maintains or even increases the level of climate-changing emissions. Therefore, the legal framework of the market, which is arguably the most powerful "soft" constraint, needs new adjustments to prevent growing unemployment, or the substitution of poorly-paid part-time jobs for well-paid full-time jobs, and pollution. Taxation of energy [2] according to its productive power, and the corresponding reduction of taxes and levies on labor, may be such an adjustment, which should also contribute to resource conservation and the preservation of social and climate stabilitiy [8]. Decision makers who aim at economic growth, social well being, and climate protection face great challenges as long as fossil fuels power substantial parts of our economies.

We propose to substantiate and extend these conclusions more rigorously within the conventional framework of modeling production and growth, taking into account the high productive power of energy through energy's output elasticity in the production function, in combination with the relevant technological and further constraints in the calculus of the representative entrepreneur on the choice of factor inputs. This implies the following procedure and research agenda: i) estimation of an energy-dependent production function (as described in Section 2), ii) estimation of the "hard" technological constraints on the utilization and automation of the capital stock (as indicated in Section 3 and the Appendix), and iii) mathematical modeling and estimation of the effect of the "soft" constraints, which include the organizational, financial, social, and legal issues that in the real world influence entrepreneurial decisions on the choice of factor inputs. It is important to note that both the "hard" technological constraints and the effects of the additional "soft" constraints have to be taken into account in order to reproduce empirical trajectories of production systems as a solution to cost minimization. This is the result of our analysis of the empirical trajectories in the cost mountain. Once the optimization problem, including hard and soft constraints, is calibrated to reproduce observed economic data by definition, the model could be used to develop scenarios of the future. The new model might also be specified in such a way as to replace conventional parametrizations in economic computable general equilibrium (CGE) models, on which policy analyses are based in many cases. The main benefit would be to take into account the empirically high importance of the factor energy in a consistent manner.

Acknowledgement
We thank Sebastian Kranz for helpful discussions.

5 Appendix: The cost mountain and its barriers

We consider an economy that produces its output $Y(t)$ at time t with the production factors capital $K(t)$, labor $L(t)$, and energy $E(t)$. As in standard economics, we assume that one can describe this output by a linearly homogeneous, macroeconomic production function $Y[K(t), L(t), E(t); t]$, which depends only on the actual inputs $K(t), L(t), E(t)$ and not on the path along which the system has arrived at these inputs. Therefore, the production function must be twice differentiable with respect to K, L, and E.[10] It may explicitly depend upon time when innovations change its technology parameters. Research into this type of production function since the early 1980s has been summarized by [8, 11].[11] It is convenient to present its results in terms of the dimensionless inputs

$$k(t) \equiv \frac{K(t)}{K_0}\ , \quad l(t) \equiv \frac{L(t)}{L_0}\ , \quad e(t) \equiv \frac{E(t)}{E_0}\ , \tag{1}$$

the dimensionless output $y(t) \equiv Y(t)/Y_0$, and the dimensionless production function

$$y[k, l, e; t] \equiv \frac{Y[kK_0, lL_0, eE_0; t]}{Y_0}\ , \tag{2}$$

where K_0, L_0, E_0, and Y_0 are the inputs and the output in a base year t_0. For the sake of simplicity we omit the time arguments of the factors. From the total differential of $y[k, l, e; t]$ one obtains the growth equation

$$\frac{\mathrm{d}y}{y} = \alpha\frac{\mathrm{d}k}{k} + \beta\frac{\mathrm{d}l}{l} + \gamma\frac{\mathrm{d}e}{e} + \delta\frac{\mathrm{d}t}{t-t_0}; \quad \delta \equiv \frac{t-t_0}{y}\frac{\partial y}{\partial t}\ . \tag{3}$$

The output elasticities of capital, α, labor, β, and energy, γ,

$$\alpha \equiv \frac{k}{y}\frac{\partial y}{\partial k},\ \beta \equiv \frac{l}{y}\frac{\partial y}{\partial l},\ \gamma \equiv \frac{e}{y}\frac{\partial y}{\partial e} \tag{4}$$

[10] The requirement of twice differentiability of thermodynamic potentials like internal energy, enthalpy and free energies leads to the Maxwell relations in thermodynamics.

[11] Objections against the concept of the aggregate macroeconomic production functions are also discussed and refuted in [8, 11].

are related to each other by the twice-differentiability condition of equal second-order mixed derivatives of $y[k, l, e, ; t]$ with respect to k, l, e :

$$l\frac{\partial\alpha}{\partial l} = k\frac{\partial\beta}{\partial k}, \quad e\frac{\partial\beta}{\partial e} = l\frac{\partial\gamma}{\partial l}, \quad k\frac{\partial\gamma}{\partial k} = e\frac{\partial\alpha}{\partial e} \quad . \tag{5}$$

If one takes into account that $\alpha+\beta+\gamma = 1$ at any fixed time t, because the production function is linearly homogenous in k, l, e, eq. (5) turns into the coupled partial differential equations for the output elasticities:

$$l\frac{\partial\alpha}{\partial l} = k\frac{\partial\beta}{\partial k}, \quad k\frac{\partial\alpha}{\partial k} + l\frac{\partial\alpha}{\partial l} + e\frac{\partial\alpha}{\partial e} = 0, \quad k\frac{\partial\beta}{\partial k} + l\frac{\partial\beta}{\partial l} + e\frac{\partial\beta}{\partial e} = 0 \quad . \tag{6}$$

Their most general solutions are

$$\alpha = A\left(\frac{l}{k}, \frac{e}{k}\right), \; \beta = B\left(\frac{l}{k}, \frac{e}{k}\right) = \int^k \frac{l}{k'}\frac{\partial A}{\partial l}\mathrm{d}k' + J\left(\frac{l}{e}\right), \; \gamma = 1 - \alpha - \beta \; ; \tag{7}$$

here $A(l/k, e/k)$ and $J(l/e)$ are any differentiable functions of their arguments. The output elasticities, and thus the combinations of k, l, e, must satisfy the restrictions

$$\alpha \geq 0, \quad \beta \geq 0, \quad \gamma = 1 - \alpha - \beta \geq 0 \; , \tag{8}$$

which result from the technical-economic requirement that all output elasticities must be non-negative. Otherwise, the increase of an input would result in a decrease of output—a situation the economic actors will avoid.

It is not hard to verify with the help of eqs. (3) and (7) that the corresponding general form of the twice-differentiable, linearly homogeneous production function is given by the r.h.s. of

$$y(t) = e\mathcal{F}\left[\frac{l}{k}, \frac{e}{k}; t\right] \equiv e\mathcal{F}\left[u, v; t\right] \; , \tag{9}$$

where $u \equiv l/k, \; v \equiv e/k$. Inserting (9) into (3) yields

$$\frac{\mathrm{d}\mathcal{F}}{\mathcal{F}} = \beta(u, v)\frac{\mathrm{d}u}{u} - \left[\alpha(u, v) + \beta(u, v)\right]\frac{\mathrm{d}v}{v} + \delta\frac{\mathrm{d}t}{t - t_0}; \; \delta \equiv \frac{t - t_0}{\mathcal{F}}\frac{\partial\mathcal{F}}{\partial t} \; . \tag{10}$$

The cost P_Y of producing a given output $Y(t) = Y[K, L, E; t]$ is obtained by multiplying the time-dependent prices per capital unit, $p_K(t)$, labor unit, $p_L(t)$, and energy unit, $p_E(t)$, by the factor quantities K, L, E required to produce that output: $P_Y = p_K K + p_L L + p_E E = P_K k + P_L l + P_E e$, where $P_K \equiv p_K K_0, \; P_L \equiv p_L L_0, \; P_E \equiv p_E E_0$.

To relate the total factor cost at time t to the output at t we rewrite it as $P_Y = e[P_K k/e + P_L l/e + P_E]$, express e by $y(t) = Y(t)/Y_0$ with the help of eq. (9), and obtain the equation for the cost mountain as

$$P_Y = \frac{Y(t)}{Y_0\mathcal{F}[u, v; t]}\left[P_K\frac{1}{v} + P_L\frac{u}{v} + P_E\right] \; . \tag{11}$$

The cost mountain rises above the $u - v$ plane, its topography is determined by the prices and ratios of the production factors, and its height scales with $Y(t)/Y_0$.

The profit G_Y obtained from the output Y is $G_Y = Y - P_Y$. Profit maximum means that

$$-G_Y = \frac{Y(t)}{Y_0\mathcal{F}(u, v; t)}\left[P_K\frac{1}{v} + P_L\frac{u}{v} + P_E\right] - Y(t) \tag{12}$$

is minimum. Since the cost mountain and the mountain of negative profit just differ by the height-shift $Y(t)$, the structure of both mountains is the same.

The negative gradient, $-\mathbf{grad} \equiv -\mathbf{e_u}\partial/\partial u - \mathbf{e_v}\partial/\partial v$, of $-G_Y$ points in the direction where the economy would move, if the representative entrepreneur at the controls would only have the one objective of profit maximization; $\mathbf{e_u}$ and $\mathbf{e_v}$ are the unit vectors parallel to the u-axis and the v-axis that span the $u-v$ plane. Operating with $-\mathbf{grad}$ on $-G_Y$ in eq. (12) and observing that eq. (10) leads to $\partial\mathcal{F}/\partial u = \beta\mathcal{F}/u$ and $\partial\mathcal{F}/\partial v = -(\alpha+\beta)\mathcal{F}/v$ one obtains the gradient of profit

$$\mathbf{grad}G_Y = -\mathbf{e_u}\frac{\partial P_Y}{\partial u} - \mathbf{e_v}\frac{\partial P_Y}{\partial v} \; , \tag{13}$$

as the negative gradient of the cost mountain with the components

$$-\mathbf{e_u}\frac{\partial P_Y}{\partial u} = -\mathbf{e_u}\frac{Y(t)}{Y_0\mathcal{F}(u,v;t)}\left[P_L\frac{1}{v} - \frac{\beta}{u}\left(P_K\frac{1}{v} + P_L\frac{u}{v} + P_E\right)\right] \; , \tag{14}$$

$$-\mathbf{e_v}\frac{\partial P_Y}{\partial v} = -\mathbf{e_v}\frac{Y(t)}{Y_0\mathcal{F}(u,v;t)}\left[\frac{\alpha+\beta}{v}\left(P_K\frac{1}{v} + P_L\frac{u}{v} + P_E\right) - (P_K + P_L u)\frac{1}{v^2}\right] \; . \tag{15}$$

To compute the topography and the gradients of the cost mountain at given factor prices one needs the output elasticities $\alpha(u,v)$ and $\beta(u,v)$ that are appropriate for the considered economy. Integration of eq. (3) or (10) with these elasticities yields the corresponding production function $e\mathcal{F}(u,v;t)$. The method of the characteristic basis curves in the theory of partial differential equations would facilitate the exact determination of $\alpha(u,v)$ and $\beta(u,v)$, if one knew them on a boundary curve and a boudary surface in k, l, e-space, respectively. This information is not, and never will be, available. Therefore, one can only calculate approximate output elasticities and production functions.

Well-known twice-differentiable macroeconomic production functions in three factors, such as capital, labor, and energy, which were designed before the eqs. (3)–(9) had been derived, are one type of approximations. Simple, and much employed, is the Cobb-Douglas production function, whose energy-dependent version is given by

$$y_{CDE} = y^0_{CDE}k^{\alpha_0}l^{\beta_0}e^{1-\alpha_0-\beta_0} = y^0_{CDE}e\left(\frac{k}{e}\right)^{\alpha_0}\left(\frac{l}{e}\right)^{\beta_0} \; . \tag{16}$$

Its output elasticities α_0 and β_0 are constants and satisfy the differential equations (6) trivially. CES functions have constant elasticities of substitution; their energy-dependent output elasticities are given in [11]. Another type of approximations are LinEx functions, which depend *lin*early on energy and *ex*ponentially on the factor ratios. Their output elasticities are required to satisfy asymptotic boundary conditions of the differential equations (6). These conditions incorporate the law of diminishing returns and the approach to the state of maximum automation [8, 11]. The simplest LinEx function is

$$y_{L1} = y^0_{L1}e\exp\left[a\left(2 - \frac{l+e}{k}\right) + ac\left(\frac{l}{e} - 1\right)\right] , \tag{17}$$

where a, c, and y^0_{L1} are technology parameters; innovations may change them in time and contribute to δ in eq. (3). The corresponding output elasticity of capital, $\alpha_{L1} = a(l+e)/k$, vanishes for vanishing ratios of labor and energy to capital and thus reflects the law of diminishing returns. The output elasticity of labor, $\beta_{L1} = a\,(cl/e - l/k)$, vanishes, when the capital stock approaches the magnitude k_m required for maximum automation, and when simultaneously the energy input approaches the quantity $e_m = ck_m$ that is demanded by the fully utilized capital stock k_m. As one can see from the growth equation (3), in this state of maximum automation an additional unit of routine labor l does not contribute to the growth of output any more. β also vanishes, if $e \to ck$ for any $k < k_m$. This imposes a limit on the region of accessible factor space that results from the restriction that output elasticities must be non-negative. The vanishing of β_{L1} is a necessary but not sufficient condition for the state of maximum automation. The additional characterization of this state is that its labor input l_m is much less than k_m.[12]

To visualize the trajectory of a real-life economy on the slope of its cost mountain the time-changing factor prices and inputs and an appropriate production function $e\mathcal{F}(u, v; t)$ are needed in eqs. (11), (14), and (15). Aggregate, deflated factor prices are available for Germany's industrial sector "Warenproduzierendes Gewerbe" (GWG) between 1960 and 1981 from [14]. The production function should reproduce economic growth reasonably well with small residuals and acceptable statistical quality measures. The LinEx function (17) satisfies these criteria. The energy-dependent Cobb-Douglas function (16) also reproduces economic growth in Germany, Japan and the USA for about two decades with not too large residuals, albeit with considerably worse statistical quality measures than the LinEx function, if its constant output elasticies are *not* chosen to be equal to the factor cost shares. Rather they, as well as the technology parameters of the LinEx function (17), have been determined econometrically, observing the restrictions (8) of non-negative output elasticities [8].

The cost mountain (11) that rises above the $u - v$ plane is shaped by factor inputs, factor costs, the production function, and the output elasticities. The projections of the trajectory of GWG and of the cost gradients onto the $u - v$ plane are shown in Fig. 2 (which is a modified update of Fig. 4 of [14]). For the sake of simplicity the projection is made for the fraction of output that is equal to Y_0; thus $Y(t)/Y_0 = 1$ in eqs. (14) and (15). The time-averaged LinEx output elasticities are about 0.45 for capital, 0.05 for labor, and 0.50 for energy and the corresponding output elasticities of the energy-dependent Cobb-Douglas function are close to these numbers [8, 14].[13]

The shadow-price barrier from the constraint that capacity utilization cannot exceed 1 is given by [8, 10]

$$1 = \eta_0 \left(\frac{l}{k}\right)^{\lambda} \left(\frac{e}{k}\right)^{\nu} \equiv \eta_0 u^{\lambda} v^{\nu}, \tag{18}$$

where the parameters λ, ν, and η_0 have been determined by fitting the r.h.s. of eq. (18) to the data on capacity utilization in the total German economy. These data, pub-

[12] A modified LinEx function $y_{L11} = y^0_{L11} e \exp\left[a\left(1 - \frac{l}{k}\right) + \frac{1}{c}\left(1 - \frac{e}{k}\right) + ac\left(\frac{l}{e} - 1\right)\right]$, has output elasticities $\alpha_{L11} = al/k + e/ck$, and $\beta_{L11} = a\,(cl/e - l/k) = \beta_{L1}$, which satify the same asymptotic boundary conditions as α_{L1} and β_{L1}. Furthermore, because of the restrictions (8), $e = ck$ is accessible only, and only asymptotically, in the state $k_m, l_m << k_m, e_m = ck_m$ of maximum automation. For $e_m = ck_m$, the growth equation (3) at fixed t, with $\gamma = 1 - \alpha_{L1(1)} - \beta_{L1(1)}$, turns into $dy/y = dk/k$.

[13] The cost gradients shown in Fig. 2 were calculated with the LinEx and Cobb-Douglas production functions of [14], whose variables are normalized to the base year 1970. In eq. (18) and Fig. 2, the base year is $t_0 = 1960$. The production functions of [8], whose variables are normalized to $t_0 = 1960$, reproduce economic growth somewhat better than those of [14].

lished by the "Sachverständigenrat für die Gesamtwirtschaft" are not necessarily identical with capacity utilization in GWG, but they are the only ones available. The parameters $\eta_0 = 1,\ \lambda = -0.152,\ \nu = 0.386$ reproduce the data satisfactorily, although the fit stays somewhat below the maxima in 1965, 1969, and 1970. The barrier from the technological constraint (18) is indicated by full squares in Fig. 2. A complete fit to the (non-available) capacity utilization data for GWG would probably move it somewhat closer to the empirical trajectory. The LinEx approximation indicates that the barrier from maximum automation is in the region where $v = c \approx 1$, so that $\beta_{L1} = 0$, *and* where $u << 1$.

We note that along all of the path the negative gradients of the cost mountain point toward the shadow-price barrier from capacity utilization and the region of maximum automation.

References

[1] Stern Review Report on the Economics of Climate Change, ISBN number: 0-521-70080-9, Cambridge University Press (http://www.cambridge.org/9780521700801) 2007. Stern, N.: The Economics of Climate Change. Amer. Econ. Rev. **98(2)**, 1–37 (2008)

[2] Murray, J., King, D.: Oil's tipping point has passed. Nature **481**, 433-435 (2012)

[3] Nordhaus, W.: A Question of Balance. Weighting the Options on Global Warming Policies. Yale University Press, New Haven & London, 2008

[4] Solow, R.M.: The Economics of Resources and the Resources of Economics. Amer. Econ. Rev. **64**, 1–14 (1974).

[5] Baron, R.: Competitive Issues Related to Carbon/Energy Taxation, Annex I Expert Group on the UN FCCC, Working Paper 14, ECON-Energy, Paris, 1997

[6] Ströbele, W.: Growth Models with Restrictions Concerning Energy Resources: An Attempt to Identify Critical Parameters and Structural Features; in: W. Eichhorn, R. Henn, K. Neumann, R. W. Shephard (Eds.), Economic Theory of Natural Resources, Physica, Würzburg, Wien, 1982; pp. 389-406

[7] Strahan, D.: The Last Oil Shock. London: John Murray, 2007

[8] Kümmel R.: The Second Law of Economics: Energy, Entropy, and the Origins of Wealth. New York, Dordrecht, Heidelberg, London: Springer (The Frontiers Collection); 2011. http://www.springer.com/physics/complexity/book/978-1-4419-9364-9.

[9] Weissmahr, J., private communication

[10] Kümmel, R., Ayres, R.U., Lindenberger, D.: Thermodynamic laws, economic methods and the productive power of energy. J. Non-Equilib. Thermodyn. **35**, 145–179 (2010); DOI 10.15.15/JNETDY.2010.009

[11] Lindenberger, D., Kümmel, R.: Energy and the state of nations. Energy **36**, 6010–6018 (2011); DOI: 10.1016/j.energy.2011.08.014

[12] Ayres, R.U., Warr, B.: Accounting for growth: the role of physical work. Struct. Change Econ. Dynam. **16**, 181–209 (2005)

[13] Ayres, R.U., Warr, B.: The Economic Growth Engine. Edgar Elgar, Cheltenham, 2009

[14] Kümmel, R., Strassl, W.: Changing energy prices, information technology, and industrial growth; in: W. van Gool, J.J.C. Bruggink (Eds.), Energy and Time in the Economic and Physical Sciences, North-Holland, Amsterdam, 1985, pp. 175-194; the aggregate German energy prices between 1960 and 1981 are given here.